Electromagnetism 2 Summary Notes

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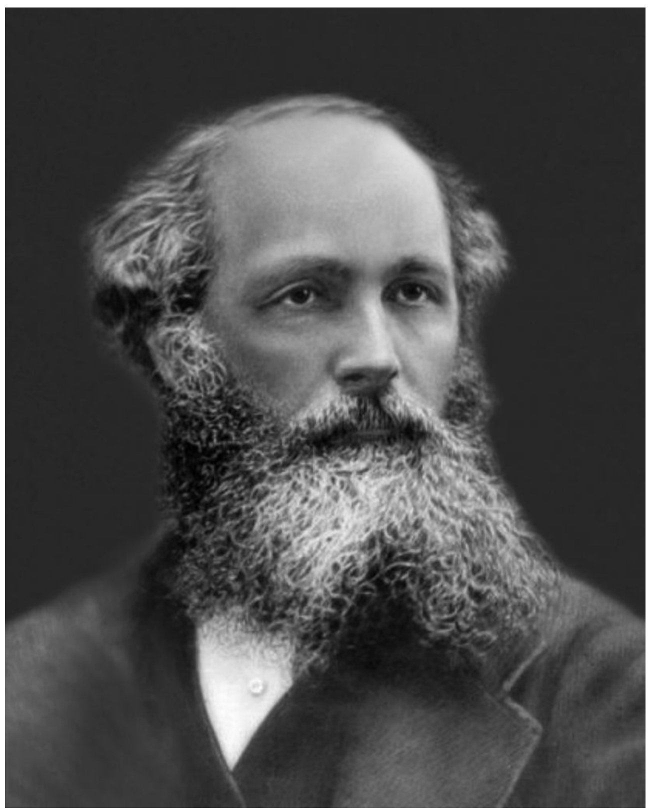

Contents

1 Maxwell's Equations

Maxwell I was derived from Coulomb's law,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}. \tag{MI}$$

Maxwell II was derived from the concept of there being no magnetic monopoles,

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{MII}$$

From Faraday's law, induced emf due to a changing magnetic flux is given by

$$\epsilon = -\frac{\partial \phi_B}{\partial t} \tag{1}$$

Maxwell III was derived from Faraday's law,

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{MIII}$$

Consider a solenoid connected to a current source, a loop passes around the solenoid and is connected to a voltmeter. The magnetic field outside the solenoid is negligible. If the magnetic field inside the solenoid varies with time, there is a potential difference measured in the loop. Conversely, we can consider the voltage induced in a loop of wire moving through a uniform magnetic field.

We use Ampere's law,

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I, \tag{2}$$

and consider a correction to it for a capacitor along a current carrying wire to derive **Maxwell IV**,

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}. \tag{MIV}$$

Helmholtz's theorem states that the divergence and curl of a vector field are sufficient to fully specify the field.

1.1 Two Useful Theorems

The Divergence theorem states

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{E}) \mathrm{d}V = \int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S}, \tag{3}$$

where V is the volume bounded by surface S.

Stoke's Theorem states

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot \mathrm{d}\boldsymbol{S} = \oint_s \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}, \tag{4}$$

where S is the surface bounded by closed path s.

Note that, when taking these integrals, it is important to consider the direction taken, otherwise you may be a factor of -1 off. Path integrals require compatible orientation between the path and the surface (see mathematical methods notes), and it is convention for surface integrals to have the vector surface element pointing out of the enclosed volume.

In addition, the divergence theorem requires the surface to be closed.

A useful vector identity is

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{a}) - \nabla^2 \boldsymbol{a} \tag{5}$$

for any vector field $\boldsymbol{a}$ sufficiently integratable.

2 Electromagnetic Waves in a Vacuum

2.1 Plane Waves

In a vacuum, Maxwell's equations lead us to

$$\begin{aligned} \nabla^2 \boldsymbol{E} &= \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \\ \nabla^2 \boldsymbol{B} &= \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \end{aligned} \tag{6}$$

These have solutions as plane waves,

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E_0} \exp i(\boldsymbol{k} \cdot \boldsymbol{r} - wt) \tag{7}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{B_0} \exp i(\boldsymbol{k} \cdot \boldsymbol{r} - wt \tag{8}$$

In a vacuum, $\boldsymbol{E}, \boldsymbol{k}, \boldsymbol{B}$ are orthogonal, $B_0 = E_0/c$, and $\boldsymbol{E_0} \times \boldsymbol{B_0}$ gives the direction of $\boldsymbol{k}$, the direction of propagation. Substituting in these solutions give the dispersion relation for light in a vacuum

$$\omega = k\sqrt{\frac{1}{\mu_0 \varepsilon_0}} \implies v_g \equiv \frac{\partial \omega}{\partial k} = \sqrt{\frac{1}{\mu_0 \varepsilon_0}} \equiv c. \tag{9}$$

2.2 The Continuity Equation

Taking the divergence of Maxwell IV, and using $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) = 0$ generally, we find the continuity equation in integral form,

$$\int \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} = -\frac{\partial}{\partial t} \int \rho \mathrm{d}V. \tag{10}$$

This is equivalent to conservation of charge, the charge flowing out through the surface per second is equal to the change in total charge inside the volume per second. Maxwell's equations *require* conservation of charge.

3 Special Relativity

Einstein considered two seemingly equivalent scenarios, relating to a current carrying wire stationary relative to the lab frame, and a positive test charge also stationary relative to the lab frame.

1. An observer P sat on a positive charge in the wire
2. An observer N sat on a negative charge moving at velocity $\boldsymbol{v}$ relative to the lab frame.

Thinking classically, the force on the test charge in frame P is 0, as the test charge is stationary, and the electric field is 0 (equal density of positive and negative charges cancel by Gauss' law),

$$\boldsymbol{F} = q(\cancel{\boldsymbol{E}} + \cancel{\boldsymbol{v}} \times \boldsymbol{B}) = 0, \tag{11}$$

so observer P does not see the test charge move.

In observer N's frame, however, the test charge has velocity $-\boldsymbol{v}$, and so feels force

$$\boldsymbol{F} = q(\cancel{\boldsymbol{E}} + \boldsymbol{v} \times \boldsymbol{B}) \neq 0. \tag{12}$$

Two assumptions led to special relativity,

1. The laws of Physics are the same in all inertial frames
2. The velocity of light is the same in all inertial frames, and is independent of the velocity of the source of the light or the observer.

Assumption 1 is not compatible with our two cases above. Einstein explained that there should be a force on the test charge in observer P's frame too, due to the length contraction of the moving negative charges, meaning the positive and negative charges are no longer evenly distributed, and there is a net electric field, the test charge feels a force pulling it towards the wire, consistent with what observer N sees.

There are to two general results of Special Relativity, for two inertial frames o and p,

$$\Delta x_o = \frac{\Delta x_p}{\gamma} \tag{13}$$

$$\Delta\tau_o = \gamma\Delta\tau_p, \tag{14}$$

where (13) is the Lorentz contraction, and (14) is the time dilation. o and p denote the length and time in the observer and proper frame, where the observer is moving at speed v relative to the proper frame, and

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{15}$$

4 Conducting and Superconducting Materials

4.1 Extensive and Intensive Properties of Conductors

Ohm's law can be written in terms of the electric field and current density,

$$\boldsymbol{J} = \sigma_n \boldsymbol{E}, \tag{16}$$

where $\sigma_n = 1/\rho_n$ is the conductivity of the material.

In general, the movement of bulk charge density in a conductor is so great that we can assume, to a good approximation, that there is no bulk charge density inside (can show this is a good approximation as the characteristic lifetime ($\tau \approx 10^{-70}s$ for copper) is much lower than the frequency of visible light).

4.2 The Drude Model

The electric field accelerates the charges which then collide with the scattering sites. The charge carriers accelerate for an average time τ, they then scatter and instantaneously stop. The fraction of charge carriers that scatter in time δt is $\delta t/\tau$, hence the momentum of the charge carriers at $t + \delta t$ is given by

$$p(t + \delta t) = (1 - \frac{\delta t}{\tau})p(t) + (1 - \frac{\delta t}{\tau})F(t)\delta t + F(t)\delta t \frac{\delta t}{\tau}, \tag{17}$$

where we have considered the initial momentum of the un-scattered charges, their additional momentum from the force, and the additional momentum to the scattered charges due to the force. Setting $\delta p = p(t + \delta t) - p(t)$, for some infinitesimal δt,

$$\frac{\partial p}{\partial t} = -\frac{p(t)}{\tau} + F(t). \tag{18}$$

The Lorentz Force is given by

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \approx q\boldsymbol{E}, \tag{19}$$

where we have used that the magnitude of the electric field is a factor of c greater than the magnetic field. From this, we find the equation of motion for charge carriers,

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q\boldsymbol{E} - \frac{m\boldsymbol{v}}{\tau}. \tag{20}$$

Note that we can interpret the second term as a frictional damping term.

Current density is given by

$$\boldsymbol{J} = \rho \boldsymbol{v} = Nq\boldsymbol{v}, \tag{21}$$

subbing this into (20) yields

$$m\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = Nq^2\boldsymbol{E} - \frac{m\boldsymbol{J}}{\tau}. \tag{22}$$

Using complex trial solutions for $\boldsymbol{J}$ and $\boldsymbol{E}$, assuming they oscillate with the same frequency, we find

$$\tilde{\boldsymbol{J}} = \sigma_n \tilde{\boldsymbol{E}} = \frac{Nq^2}{m(\tau^{-1} - i\omega)} \tilde{\boldsymbol{E}}, \tag{23}$$

where tilde indicates complex fields. We can simplify this to find the real component of $\boldsymbol{J}$,

$$\boldsymbol{J} = \frac{Nq^2}{m\sqrt{1 + (\omega\tau)^2}} \cos\left(\omega(t - \tau)\right) \boldsymbol{E_0}, \tag{24}$$

the current density lags the $\boldsymbol{E}$-field by $\omega\tau$.

4.3 Dispersive and Ballistic Motion of Waves

A wave-packet is made up of many component plane waves. Each component plane wave has its own wave-vector and phase velocity,

$$k_{\text{real}} = \frac{2\pi}{\lambda} \quad \text{and} \quad v_{\text{phase}} = \frac{\omega}{k_{\text{real}}}. \tag{25}$$

If we consider a wave packet built with a Gaussian distribution of wave-vectors, where k_0 is the mean wave-vector, we can Taylor expand the dispersion relation,

$$\omega = f(k) = \omega_0 + \alpha(k - k_0) + \beta/2(k - k_0)^2 + ... \tag{26}$$

where

$$\alpha = \left.\frac{\partial \omega}{\partial k_{\text{real}}}\right|_{k=k_0} \quad \text{and} \quad \beta = \left.\frac{\partial^2 \omega}{\partial k_{\text{real}}^2}\right|_{k=k_0}. \tag{27}$$

and ω_0 is the angular frequency of the 'most important' component wave.

To second order, we get a solution whose peak moves through space as a function of time,

$$x_{\text{peak}} = \alpha t, \tag{28}$$

therefore we can equate α to the group velocity of the whole wave-packet,

$$v_{\text{group}} = \alpha = \left.\frac{\partial \omega}{\partial k_{\text{real}}}\right|_{k=k_0}. \tag{29}$$

Waves are ballistic (wave-packet moves as one) when

$$\beta = \left.\frac{\partial^2 \omega}{\partial k_{\text{real}}^2}\right|_{k=k_0} = 0, \tag{30}$$

and waves are dispersive (wave-packet spreads out over time) when

$$\beta = \left.\frac{\partial^2 \omega}{\partial k_{\text{real}}^2}\right|_{k=k_0} \neq 0. \tag{31}$$

4.4 Electromagnetic Waves Propagating Through Metals

In metals, the scattering time is short (compared to $1/f$ of the EM waves). In the limit $\tau \to 0$, (23) becomes

$$\tilde{\boldsymbol{J}} = \frac{Nq^2}{m(\tau^{-1} - i\omega)}\tilde{\boldsymbol{E}} = \frac{Nq^2\tau}{m}\tilde{\boldsymbol{E}} \implies \sigma_n = \frac{Nq^2\tau}{m}. \tag{32}$$

Maxwell I for a conducting material is

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0. \tag{33}$$

Taking the curl of Maxwell III, substituting Ohm's law into Maxwell IV and combining gives

$$\nabla^2 \boldsymbol{E} = \mu_0 \sigma_n \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \tag{34}$$

which is solved by a wave travelling in the x-direction. Inserting a trial solution, we find

$$k^2 = \mu_0 \varepsilon_0 \omega^2 + i\omega\mu_0\sigma_n, \tag{35}$$

so k is complex,

$$k \equiv k_{\text{real}} + ik_{\text{im}}. \tag{36}$$

The real component k_{real} represents oscillations, the imaginary component k_{im} represents decay, to understand this consider the form of the plane wave solution,

$$\boldsymbol{E} = \boldsymbol{E}_0 \mathrm{e}^{i(kx-\omega t)} = \boldsymbol{E}_0 \mathrm{e}^{i(k_{\text{real}}x-\omega t)}\mathrm{e}^{-k_{\text{im}}x}. \tag{37}$$

4.4.1 Dispersion Relation for a Highly Insulating Material

For a highly insulating material, $\sigma_n \to 0$, so $\mu_0\varepsilon_0\omega^2 \gg \mu_0\sigma_n\omega$ and we find the dispersion relation to be

$$k = \omega\sqrt{\mu_0\varepsilon_0}. \tag{38}$$

Hence, ω is linear in k and the wave is ballistic.

4.4.2 Dispersion Relation for a Highly Conducting Material

For a highly conducting material, $\sigma_n \to \infty$, so the inverse is true, and we find the dispersion relation to be

$$k \equiv k_{\text{real}} + ik_{\text{im}} = (1+i)\sqrt{\frac{\omega\mu_0\sigma_n}{2}} \tag{39}$$

We find the solution for the $\boldsymbol{E}$-field inside a good conductor by substituting k back into the plane wave equation, we take the real (or imaginary, they just differ in phase) component, giving

$$\boldsymbol{E} = \boldsymbol{E_0} \cos\left(\frac{x}{\delta} - \omega t\right) \exp\left(-\frac{x}{\delta}\right), \tag{40}$$

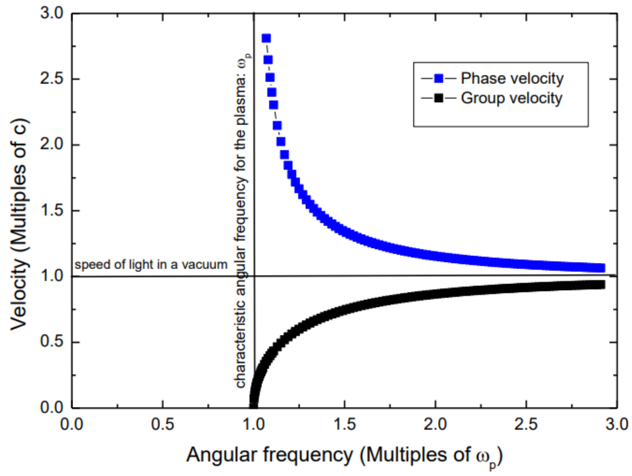

Figure 1: The velocity of an electromagnetic wave in a plasma as a function of angular frequency.

where

$$\delta \equiv \frac{1}{k_{\text{real}}} = \frac{1}{k_{\text{im}}} = \sqrt{\frac{2}{\omega \mu_0 \sigma_n}}. \tag{41}$$

The exponential part causes the field to decay, showing that δ is a decay length, known as the *skin depth* for a good conductor.

4.5 Electromagnetic Waves Propagating Through Low Density Plasmas

In low density plasmas, scattering time τ is long, so $w \gg \tau^{-1}$. (23) becomes

$$\sigma_n = \frac{iNq^2}{m_e \omega}, \tag{42}$$

and the dispersion relation can therefore be written

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \left(\frac{\omega_p}{\omega}\right)\right), \tag{43}$$

where the angular plasma frequency is given by

$$\omega_p = \left(\frac{Ne^2}{m_e \varepsilon_0}\right)^{\frac{1}{2}}. \tag{44}$$

This has two distinct regimes,

1. $\omega > \omega_p$: k is real and the EM waves propagate without attenuation throughout the plasma
2. $\omega < \omega_p$: k is imaginary and we get attenuation due to the negative exponential term

Note the similarity with metals, in metals the reflectivity is constant (very high) until ω_p, the plasma frequency, when it drops to near zero. In plasmas, waves below ω_p are attenuated.

We can find the phase and group velocities as usual, see Fig. 1. Phase velocity can be greater than c, as energy does not travel at the phase velocity.

4.6 Superconducting Materials

Superconductors are materials which, at temperatures below the *critical temperature* T_C have zero resistivity, we can think of them as having electrons condensed into a superconducting ground state. A superconductor can be modelled as an enormous atom with a macroscopic wavefunction.

4.6.1 The First London Equation

If we assume electrons are not scattered, they each feel a force

$$\boldsymbol{F} = m_e \dot{\boldsymbol{v}} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \approx q\boldsymbol{E}, \tag{45}$$

using $\boldsymbol{J} = Ne\boldsymbol{v}$, we find

$$\boldsymbol{E} = \frac{m_e}{N_s e^2} \frac{\partial \boldsymbol{J}}{\partial t} = \mu_0 \lambda_L^2 \frac{\partial \boldsymbol{J}}{\partial t}, \tag{L1}$$

where $\lambda_L^2 = \frac{m_e}{\mu_0 N_s e^2}$, and N_s is the density of superelectrons.

4.6.2 Second London Equation

Taking the curl of (L1), we find

$$\boldsymbol{B} = -\mu_0 \lambda_L^2 \boldsymbol{\nabla} \times \boldsymbol{J}. \tag{L2}$$

Substituting Maxwell IV for $\boldsymbol{J}$, where $\partial \boldsymbol{E}/\partial t = 0$, and rearranging, we find

$$\nabla^2 \boldsymbol{B} = \frac{1}{\lambda_L^2} \boldsymbol{B}, \tag{46}$$

which, for propagation in a semi-infinite slab starting at $x = 0$, has solutions of the form

$$\boldsymbol{B}(x) = \boldsymbol{B_0} \exp(-x/\lambda_L) \quad \text{for} \quad x > 0, \tag{47}$$

which describes the *Meissner state*: the magnetic field is excluded from the bulk of all metallic superconductors in low applied fields.

5 Dielectrics

5.1 Polarization

In dielectrics, all charges are attached to specific atoms or molecules, so all they can do is move a bit within the atom or molecule. Applying an electric field $\boldsymbol{E}$ to a dielectric influences the charge within the atoms, *polarizing* the atoms, such that each atom now has a tiny electric dipole moment $\boldsymbol{p}$,

$$\boldsymbol{p} \equiv q\boldsymbol{d}, \tag{48}$$

where $\boldsymbol{d}$ is the vector from the negative to the positive charge (points in the same direction as the electric field).

The polarization of a dielectric, $\boldsymbol{P}$, is the electric dipole moment per unit volume,

$$\boldsymbol{P} \equiv N\boldsymbol{p}, \tag{49}$$

where N is the number of electric dipoles per unit volume.

A polar dielectric contains randomly oriented permanent dipole moments, charge distribution is asymmetric. An applied magnetic field aligns these dipoles. A non-polar dielectric contains a symmetrical charge distribution, the applied field causes charge separation.

5.2 The Field of a Polarized Object

The potential, and hence the field, of a polarized object is the same as is produced by a surface charge density

$$\sigma_b \equiv \boldsymbol{P} \cdot \hat{\boldsymbol{n}}, \tag{50}$$

plus a volume charge density

$$\rho_b \equiv -\boldsymbol{\nabla} \cdot \boldsymbol{P}. \tag{51}$$

We can therefore consider a polarized object to be a distribution of *bound charges*; at the surface, bound charges can be though of as the ends of polar molecules, in the volume, they can be thought of as an accumulation of the ends of molecules due to a non-uniform polarization.

If the polarisation changes with time, the moving charges are equivalent to a current. The current density is given by

$$\boldsymbol{J}_b = \frac{\partial \boldsymbol{P}}{\partial t} \tag{52}$$

5.3 The Electric Displacement Field

Within a dielectric, the total charge density can be written

$$\rho = \rho_b + \rho_f, \tag{53}$$

where ρ_f is free charge density (any charge that is not a result of polarization).

We can write Maxwell I,

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_b+\rho_f}{\varepsilon_0} = \frac{-\boldsymbol{\nabla}\cdot\boldsymbol{P}+\rho_f}{\varepsilon_0}, \tag{54}$$

where $\boldsymbol{E}$ is the total field. We can combine the two divergence terms, and find

$$\boldsymbol{\nabla}\cdot(\varepsilon_0\boldsymbol{E}+\boldsymbol{P}) = \rho_f, \tag{55}$$

we define the expression in parentheses as $\boldsymbol{D}$,

$$\boldsymbol{D} \equiv \varepsilon_0\boldsymbol{E}+\boldsymbol{P}, \tag{56}$$

this is known as the *electric displacement field*. We can therefore rewrite Maxwell I as

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_f. \tag{57}$$

This is useful as it allows us to work with ρ_f, which we can control.

The displacement current density is defined by

$$\boldsymbol{J}_d \equiv \frac{\partial\boldsymbol{D}}{\partial t} = \varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}+\frac{\partial\boldsymbol{P}}{\partial t}. \tag{58}$$

5.4 Electric Susceptibility and Relative Permittivity

Linear dielectrics obey the relation

$$\boldsymbol{P} = \varepsilon_0\chi_e\boldsymbol{E}, \tag{59}$$

where χ_e is the *electric susceptibility* of the medium. In such dielectrics, we can consider the displacement field, $\boldsymbol{D}$, which can be deduced directly from the free charge distribution,

$$\boldsymbol{D} = \varepsilon_0\boldsymbol{E}+\boldsymbol{P} = \varepsilon_0(1+\chi_e)\boldsymbol{E} = \varepsilon\boldsymbol{E}, \tag{60}$$

where $\varepsilon \equiv \varepsilon_0\varepsilon_r$, and $\varepsilon_r \equiv 1+\chi_e$.

Relative permittivity is complex, with out of phase real and imaginary components.

6 Magnetic Materials

A material is *magnetized* when a magnetic field is applied to it, causing a net alignment of magnetic dipoles. Some materials acquire a magnetisation parallel to $\boldsymbol{B}$ (paramagnets), and some opposite to $\boldsymbol{B}$ (diamagnets). Ferromagnets retain their magnetization even after the external field has been removed.

Consider a magnetic dipole to be a current loop with area S and current I, the magnetic dipole moment $\boldsymbol{m}$ is given by

$$\boldsymbol{m} = IS\hat{\boldsymbol{n}}, \tag{61}$$

where $\hat{\boldsymbol{n}}$ is the vector normal to the surface S.

The magnetic dipole moment per unit volume is the magnetization, $\boldsymbol{M}$, and is given by

$$\boldsymbol{M} = N\boldsymbol{m}, \tag{62}$$

where N is the number of magnetic dipoles per unit volume.

Considering the vector potential of a single dipole moment, we can derive the total potential contributions of two currents, surface and volume. The surface current density is given by

$$\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}}, \tag{63}$$

and the volume current density by

$$\boldsymbol{J}_b = \boldsymbol{\nabla} \times \boldsymbol{M}. \tag{64}$$

This means that the potential (and hence also the field) of a magnetized object is the same as would be produced by a volume current $\boldsymbol{J}_b$ through the material, plus a surface current $\boldsymbol{K}_b$ on the boundary.

6.1 The Magnetic Field Strength

We can consider the current in a material to be the sum of the bound currents on the surface and in the volume, as derived above, and free current,

$$\boldsymbol{J} = \boldsymbol{J}_b + \boldsymbol{J}_f. \tag{65}$$

In view of this, we can write Ampere's law as

$$\frac{1}{\mu_0}(\boldsymbol{\nabla} \times \boldsymbol{B}) = \boldsymbol{J} = \boldsymbol{J}_b + \boldsymbol{J}_f = (\boldsymbol{\nabla} \times \boldsymbol{M}) + \boldsymbol{J}_f, \tag{66}$$

collecting together the curls, we find

$$\boldsymbol{\nabla} \times \left(\frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}\right) = \boldsymbol{J}_f. \tag{67}$$

We define the quantity in parenthesis as the magnetic field strength,

$$\boldsymbol{H} \equiv \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}. \tag{68}$$

We can now rewrite Maxwell IV in terms of $\boldsymbol{D}$, $\boldsymbol{H}$, and ρ_f,

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{H} &= \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t} \\ &= \boldsymbol{J}_f + \boldsymbol{J}_d. \end{aligned} \tag{69}$$

Essentially, $\boldsymbol{H}$ is useful as it allows us to consider the field due to the free charges alone.

6.2 Magnetization in Linear Media

For linear media, we can relate the magnetisation to the magnetic field strength in terms of the magnetic susceptibility,

$$\boldsymbol{M} = \chi_m \boldsymbol{H}. \tag{70}$$

A little manipulation leads us to another relation,

$$\boldsymbol{B} = \mu_0(1+\chi_m)\boldsymbol{H} = \mu\boldsymbol{H}, \tag{71}$$

where we have defined

$$\mu \equiv \mu_0\mu_r, \quad \text{and} \quad \mu_r \equiv 1+\chi_m, \tag{72}$$

μ is the *permeability* of the material.

6.3 Alternative Form of Maxwell's Equations

We can rewrite Maxwell's equations using these new definitions,

$$\nabla \cdot \boldsymbol{D} = \rho_f \tag{73}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{74}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{75}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t} \tag{76}$$

7 Radiation and Communication

7.1 Poynting's Vector

The instantaneous power per unit area for electromagnetic waves is given by Poynting's vector, $\boldsymbol{N}$, where

$$\boldsymbol{N} = \boldsymbol{E} \times \boldsymbol{H}. \tag{77}$$

We find the total instantaneous power flowing through an arbitrary surface bounding the source of the radiation by integrating Poynting's vector over the surface.

Time averaged power per unit area for a periodic field is found (shockingly) by time averaging Poynting's vector,

$$\boldsymbol{N}_{\text{avg}} = \frac{1}{T}\int_{t_0}^{t_0+T} \boldsymbol{N}\mathrm{d}t, \tag{78}$$

where $T = 2\pi/\omega$ gives the period of the field.

7.2 Potentials

We define the magnetic vector potential, $\boldsymbol{A}$, as

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{79}$$

We define the electrostatic potential, V, as

$$\boldsymbol{E} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{80}$$

which reduces to

$$\boldsymbol{E} = -\boldsymbol{\nabla} V \tag{81}$$

in time-independent situations.

Note that, by Helmholtz's Theorem, $\boldsymbol{A}$ is not fully specified by its definition, we can add to it any vector whose curl is 0. Similarly, we can add to V any function whose gradient is 0. We impose the Lorentz condition to constrain them,

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}. \tag{82}$$

We can rewrite Maxwell's equations in terms of these potentials, and each satisfy their own wave equation,

$$-\nabla^2 V + \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = \frac{\rho}{\varepsilon_0}, \tag{83}$$

$$-\nabla^2 \boldsymbol{A} + \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \mu_0 \boldsymbol{J}. \tag{84}$$

8 Hertzian Dipole

8.1 Near and Far Electromagnetic field

Consider an alternating electric dipole moment equivalent to a length $\boldsymbol{\delta l}$ of wire in which a time dependent alternating current $I(t)$ is flowing.

$$I(t)\boldsymbol{\delta l} = \omega \boldsymbol{p}(t). \tag{85}$$

From electrostatics, we know $-\boldsymbol{\nabla} V = \boldsymbol{E}$, so we know the solution for

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = -\nabla^2 V = \frac{\rho}{\varepsilon_0} \tag{86}$$

is

$$V = \frac{Q}{4\pi\varepsilon_0 r}. \tag{87}$$

We can start to solve the complex wave equation for $\tilde{\boldsymbol{A}}$ by considering the static solutions to

$$-\nabla^2 \tilde{\boldsymbol{A}} = \mu_0 \tilde{\boldsymbol{J}}, \tag{88}$$

by inspection, we find

$$\tilde{\boldsymbol{A}} = \frac{\mu_0 I_0 \delta \boldsymbol{l}}{4\pi r} \mathrm{e}^{i(kr-\omega t)}, \tag{89}$$

where $\tilde{I} = I_0 \exp(-i\omega t)$.

Changing to spherical coordinates, and using $\tilde{\boldsymbol{B}} = \boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}$, we find an expression for the magnetic field,

$$\tilde{\boldsymbol{B}}(r,\theta,\phi,t) = \frac{\mu_0 I_0 \delta l}{4\pi r^2}(1 - ikr)\sin(\theta)\mathrm{e}^{i(kr-\omega t)}\hat{\boldsymbol{\phi}}. \tag{90}$$

In the near field, $kr \ll 1$, and we get real fields

$$\boldsymbol{B}_{\text{near}}(r,\theta,\phi,t) = \frac{\mu_0 I_0 \delta l}{4\pi r^2}\sin(\theta)\cos(kr-\omega t)\hat{\boldsymbol{\phi}}. \tag{91}$$

In the far field, $kr \gg 1$, and we get

$$\boldsymbol{B}_{\text{far}}(r,\theta,\phi,t) = \frac{\mu_0 I_0 \delta l}{4\pi r} k \sin(\theta)\sin(kr-\omega t)\hat{\boldsymbol{\phi}}. \tag{92}$$

Substituting a plane wave solution for $\tilde{\boldsymbol{E}}$ into Maxwell IV, setting $\boldsymbol{J}$ to be 0, we find

$$\tilde{\boldsymbol{E}} = \frac{i}{\mu_0 \varepsilon_0 \omega} \boldsymbol{\nabla} \times \tilde{\boldsymbol{B}}, \tag{93}$$

giving us an expression for the electric field,

$$\tilde{\boldsymbol{E}} = \frac{\mu_0 I_0 \delta l}{2\pi} \frac{\omega(i + kr)\cos(\theta)}{k^2 r^3} \mathrm{e}^{i(kr-\omega t)}\hat{\boldsymbol{r}} + \frac{\mu_0 I_0 \delta l}{4\pi} \frac{\omega(i + kr - ik^2 r^2)\sin(\theta)}{k^2 r^3} \mathrm{e}^{i(kr-\omega t)}\hat{\boldsymbol{\theta}}. \tag{94}$$

This has real near and far field solutions given by

$$\boldsymbol{E}_{\text{near}} = -\frac{\mu_0 I_0 \delta l}{2\pi r^3} \frac{\omega}{k^2} \cos(\theta)\sin(kr-\omega t)\hat{\boldsymbol{r}} - \frac{\mu_0 I_0 \delta l}{4\pi r^3} \frac{\omega}{k^2} \sin(\theta)\sin(kr-\omega t)\hat{\boldsymbol{\theta}}, \tag{95}$$

$$\boldsymbol{E}_{\text{far}} = \frac{\mu_0 I_0 \delta l}{4\pi r} \omega \sin(\theta)\sin(kr-\omega t)\hat{\boldsymbol{\theta}}. \tag{96}$$

In the near field, EM waves originate from the quasi-static field produced by the moving charge. $\boldsymbol{E}_{\text{near}}$ and $\boldsymbol{B}_{\text{near}}$ are $\pi/2$ out of phase as they reach their maxima when charge separation is highest (this occurs when current is 0) and when current is highest respectively.

In the far field, both fields are tangential to the surface of a sphere centred on the dipole, they are associated with the acceleration of the charges.

8.2 Radiation and Antennae

The oscillatory nature of the dipole produces packets of EM radiation propagating away from it. They point in the direction of Poynting's vector, and carry energy away.

Beam width is defined as the angle between half-power directions. The angular dependency of the instantaneous radiated power is given by the magnitude of Poynting's vector evaluated from our expressions for $\boldsymbol{E}_{\text{far}}$ and $\boldsymbol{B}_{\text{far}}$, for time averaged power simply take the time average.

There are two types of antennae, electric and magnetic dipole antennae.

An electric dipole antenna is a length of conducting (often copper) wire connected to a voltmeter, incident EM radiation has an alternating electric field which induces an alternating current in the antenna.

A magnetic dipole antenna is a loop of conducting (again, often copper) wire connected to a voltmeter. The alternating magnetic flux through the loop due to the magnetic field of the radiation induces an alternating current in the loop.

9 Static Fields and Electromagnetic Waves Crossing Interfaces

9.1 Static Boundary Conditions Across the Interfacial Plane Between Two Dielectrics

For an insulating dielectric material, Maxwell I can be written in terms of the displacement field,

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f. \tag{97}$$

Integrating both sides over the volume of a cylinder on the border between two dielectrics, with one face on each side, we have

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{D} \mathrm{d}V = 0, \tag{98}$$

where we have used $\rho_f = 0$ in an insulating dielectric.

Applying the divergence theorem, and considering the displacement field on each surface, we have

$$\int_{S_1} \boldsymbol{D}_1 \cdot \mathrm{d}\boldsymbol{S} + \int_{S_2} \boldsymbol{D}_2 \cdot \mathrm{d}\boldsymbol{S} = 0, \tag{99}$$

where the two surfaces are the two faces. This implies that the components of the displacement field orthogonal to the interfacial plane must be equal in each medium,

$$D_{1\perp} = D_{2\perp}. \tag{100}$$

Similarly, taking the surface integral of Maxwell III over the surface S bounded by a rectangular path on the interface of the two surfaces gives

$$\int_S \boldsymbol{\nabla} \times \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}. \tag{101}$$

Rewriting with Stoke's theorem, we find

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\partial}{\partial t} \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}. \tag{102}$$

Taking the limit that the height of the path tends to 0, the magnetic flux through the rectangle tends to 0 and the only non-0 terms in the path integral are those along the two long lengths of the rectangle, giving

$$E_{1//} = E_{2//}, \tag{103}$$

equivalently saying that, for the interface between two dielectrics, the **displacement field orthogonal to the interfacial pane, and the electric field parallel to the interfacial plane are continuous**.

9.2 Static Boundary Conditions Across the Interfacial Plane Between Two Magnetic Materials

Following a similar process to the above, we find Maxwell's equations require that **the magnetic field orthogonal to the interfacial plane, and the field strength parallel to the interfacial plane, are continuous.**

$$B_{1\perp} = B_{2\perp}. \tag{104}$$

$$H_{1//} = H_{2//}, \tag{105}$$

Essentially, we can easily derive these boundary conditions by considering which fields will give a divergence or curl that is 0 at the boundary, then integrating and applying the Divergence or Stoke's theorem in the limit that the rectangle/cylinder's height goes to 0.

10 Optics

10.1 The Laws of Geometric Optics

An electromagnetic wave in medium 1 incident on medium 2 is partially reflected and partially transmitted. We can express the incident, reflected and transmitted waves generally as

$$\boldsymbol{E}_i(\boldsymbol{r}, t) = \boldsymbol{E}_{0i} \exp i(\boldsymbol{k}_i \cdot \boldsymbol{r} - \omega_i t) \tag{106}$$

$$\boldsymbol{E}_r(\boldsymbol{r}, t) = \boldsymbol{E}_{0r} \exp i(\boldsymbol{k}_r \cdot \boldsymbol{r} - \omega_r t) \tag{107}$$

$$\boldsymbol{E}_t(\boldsymbol{r}, t) = \boldsymbol{E}_{0t} \exp i(\boldsymbol{k}_t \cdot \boldsymbol{r} - \omega_t t), \tag{108}$$

where $\boldsymbol{r}$ is any position vector along the intersection of the incidence and interfacial planes.

The boundary conditions we derived above in (103) lead to

$$E_{i//} + E_{r//} = E_{t//} \tag{109}$$

at all points in the interfacial plane at all t.

Evaluating at $r = 0$, we find that
$$\omega_i = \omega_r = \omega_t. \tag{110}$$

Evaluating at $t = 0$, we find from considering reflection that
$$k_{i//} = k_{r//}, \tag{111}$$

and that
$$\theta_i = \theta_r, \tag{112}$$

this is the 1st law of geometric optics.

Considering transmission, we find the 2nd law, Snell's law
$$n_i \sin\theta_i = n_t \sin\theta_t, \tag{113}$$

and the 3rd law,
$$k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t. \tag{114}$$

10.2 Fresnel's Equations

Quantify the fraction of the $\boldsymbol{E}$-field that is transmitted and the fraction that is reflected by spitting the incident field into components normal and parallel to the plane of incidence.

10.2.1 Normal to the plane of incidence

Applying the boundary conditions for $\boldsymbol{H}$ and $\boldsymbol{E}$, we find
$$\frac{E_{0r}}{E_{0i}} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} \tag{115}$$
$$\frac{E_{0t}}{E_{0i}} = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}. \tag{116}$$

10.2.2 Parallel to the plane of incidence

$$\frac{E_{0r}}{E_{0i}} = \frac{n_i \cos\theta_t - n_t \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i} \tag{117}$$
$$\frac{E_{0t}}{E_{0i}} = \frac{2n_i \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i}. \tag{118}$$

10.2.3 The Brewster Angle and Critical Angle

At the Brewster angle, the reflected wave is fully polarised normal to the plane of incidence, the reflection coefficient parallel to the plane of incidence is zero,

$$\theta_B = \arctan \frac{n_t}{n_i} \tag{119}$$

At and above the critical angle, there is only total internal reflection of the wave,

$$\theta_C = \arcsin \frac{n_t}{n_i} \tag{120}$$

10.3 Coefficients of Reflection and Transmission

For a non-magnetic non-conducting dielectric, we use Poynting's vector to calculate the time-averaged power per unit area incident on the interface,

$$N_i = \frac{1}{2} E_{0i} H_{0i} \cos\theta_i = \frac{1}{2} \left(\frac{\varepsilon_1}{\mu_0}\right)^{\frac{1}{2}} E_{0i}^2 \cos\theta_i, \tag{121}$$

note that the factor of $\cos\theta_i$ occurs because the interface is not in general perpendicular to the direction the wave is travelling (this affects the power per unit *area*).

Similarly, for the reflected wave,

$$N_r = \frac{1}{2} \left(\frac{\varepsilon_1}{\mu_0}\right)^{\frac{1}{2}} E_{0r}^2 \cos\theta_r, \tag{122}$$

and for the transmitted wave,

$$N_t = \frac{1}{2} \left(\frac{\varepsilon_2}{\mu_0}\right)^{\frac{1}{2}} E_{0t}^2 \cos\theta_t. \tag{123}$$

The reflection and transmission coefficients are defined as the fraction of incident power that is reflected or transmitted, respectively, so

$$R \equiv \frac{N_r}{N_i} = \left(\frac{E_{0r}}{E_0 i}\right)^2, \tag{124}$$

$$T \equiv \frac{N_t}{N_i} = \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{\frac{1}{2}} \frac{E_{0t}^2}{E_{0i}^2} \frac{\cos\theta_t}{\cos\theta_i} = \frac{n_2}{n_1} \frac{E_{0t}^2}{E_{0i}^2} \frac{\cos\theta_t}{\cos\theta_i}. \tag{125}$$

By conservation of energy,

$$R + T = 1. \tag{126}$$

Fresnel's equations show that, for *highly conducting materials*, reflection coefficients are high.

11 Waveguides

Using electromagnetic waves in waveguides is a very efficient way of transmitting large amounts of power. Waveguides are hollow conducting tubes made of highly conducting materials - often copper, but in some cases superconductors.

We consider a hollow rectangular waveguide (for simplicity) carrying a transverse electric wave. Transverse electromagnetic waves *cannot* exist in hollow waveguides.

If the waveguide is a good conductor, we can assume that for steady state (no dissipation) solutions, there is no $\boldsymbol{E}$-field parallel to the edges of the waveguide. From Maxwell's equations, and this boundary condition, we can derive the dispersion relation,

$$k^2 = \frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2}. \tag{127}$$

As with plasma, we identify a lowest **cut-off** angular frequency (where $k = 0$), given by

$$\omega_c = \frac{\pi c}{a}, \tag{128}$$

below this angular frequency waves are reflected off the waveguide and absorbed by the conducting walls - k is imaginary and the wave is attenuated.

The magnetic field associated with the electric field has both transverse and longitudinal components (consider the wave bouncing off the walls).